PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Geometry Problems: *Circles and Triangles*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Geometry Problems: Circles and Triangles

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to use geometric properties to solve problems. In particular, the lesson will help you identify and help students who have the following difficulties:

- Solving problems by determining the lengths of the sides in right triangles.
- Finding the measurements of shapes by decomposing complex shapes into simpler ones.

The lesson unit will also help students to recognize that there may be different approaches to geometrical problems, and to understand the relative strengths and weaknesses of those approaches.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
6. Attend to precision.
7. Look for and make use of structure.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics:*

G-SRT: Understand similarity in terms of similarity transformations.
Define trigonometric ratios and solve problems involving right triangles.
G-C: Understand and apply theorems about circles.

INTRODUCTION

- Before the lesson, students attempt the problem individually. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students work collaboratively in small groups to produce an improved solution to the same problem.
- Working in the same small groups, students comment on and evaluate some solutions produced by students in another class.
- In a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students review the work they did on their individual solutions and write about what they learned.

MATERIALS REQUIRED

- Each individual student will need a copy of the task sheet *Circles and Triangles*, a ruler, calculator, pencil, mini-whiteboard, pen and eraser.
- Each small group of students will need a copy of *Sample Responses to Discuss,* and a large sheet of paper for making a poster.
- There are also some slides to help with instructions and to support whole-class discussion.

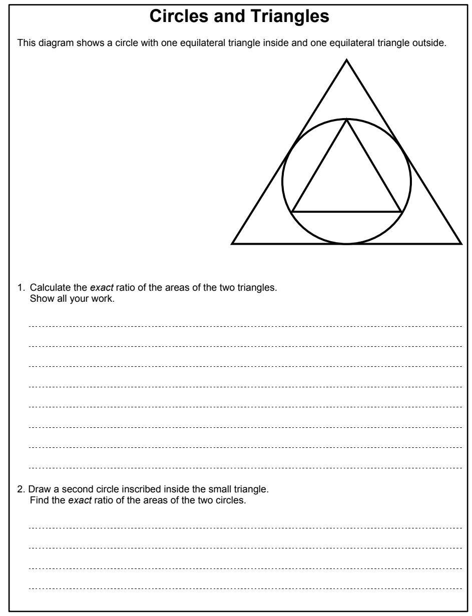
TIME NEEDED

15minutes before the lesson, a 1-hour lesson, and 10 minutes in a follow-up lesson (or for homework). Timings are approximate and will depend on the needs of the class.

BEFORE THE MAIN LESSON

Assessment task: *Circles and Triangles* (15 minutes)

Have students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.



Circles and Triangles

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

1. Calculate the *exact* ratio of the areas of the two triangles.
   Show all your work.

2. Draw a second circle inscribed inside the small triangle.
   Find the *exact* ratio of the areas of the two circles.

Give out the task *Circles and Triangles*, a pencil, and a ruler. Issue calculators if students ask for them.

Introduce the task briefly and help the class to understand the problem and its context.

> *Read through the task and try to answer it as carefully as you can.*
>
> *Show all your work so that I can understand your reasoning.*
>
> *Don't worry too much if you don't understand everything, because there will be a lesson [tomorrow] using this task.*

It is important that students are allowed to answer the questions without assistance, as far as possible. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason we suggest that, when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

Assessing students' responses

We suggest that you do not write scores on students' work. The research shows that this is counterproductive, as it encourages students to compare scores, and distracts their attention from what they might do to improve their mathematical work.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest that you write your own lists of questions, based on your own students' work, using these ideas. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will help the majority of students. These can then be written on the board at the beginning of the lesson.

Common issues:	**Suggested questions and prompts:**
Student has difficulty getting started	• What do you know about the angles or lines in the diagram? How can you use what you know? What do you need to find out? • You may find it helps to give a name to some of the lengths. Try r for the radius of the circle, x for the side of the big triangle, and so on. • Can you add any helpful construction lines to your diagram? What do you know about these lines? • Can you find relationships between the lengths from what you know about geometry?
Student works out the ratio by measuring the dimensions of the triangles	• What are the advantages/disadvantages of your method? • Are your measurements accurate enough? How do you know?
Student does not explain the method clearly **For example: The student does not explain why triangles are similar.** Or: The student does not explain why triangles are congruent.	• Would someone unfamiliar with your type of solution easily understand your work? • How do you know these triangles are similar/congruent? • It may help to label points and lengths in the diagram.
Student has problems recalling standard ratios The student recalls incorrectly or makes an error using the special ratios for a 30°, 60°, 90° triangle $(1, \sqrt{3}, 2)$.	• What do you know about cos 30°? What do you know about sin 30°? How can you use this information? • Use the Pythagorean Theorem to check/calculate the ratio of the sides of the triangle.
Student uses perception alone to calculate the ratio For example: The student rotates the small triangle about the center of the circle and assumes that the diagram alone is enough to show the ratio of areas is 4:1.	• What math can you use to justify your answer?
Student makes a technical error For example: The student makes an error manipulating an equation.	• Check to see if you have made any algebraic errors.

Common issues:	Suggested questions and prompts:
Student uses ratios of lengths rather than ratios of areas For example: When finding the ratio of the areas of the two circles, the student obtains an incorrect answer because they find the ratio of the radii, rather than the ratio of the squares of the radii.	• What is the formula for the area of the circle? How can you use it to find the ratio of the areas of the circles?
Student produces correct solutions	• Can you solve the problem using a different method? Which method do you prefer? Why?

SUGGESTED LESSON OUTLINE

Improve individual solutions to *Circles and Triangles* (10 minutes)

Return students' papers and give each student a mini-whiteboard, pen and eraser.

Recall what we were looking at in a previous lesson. What was the task?

I have read your solutions and I have some questions about your work.

If you have not added questions to individual pieces of work, write your list of questions on the board, and ask students to select questions appropriate to their own work.

Ask students to spend a few minutes answering your questions. It is helpful if they do this using mini-whiteboards, so that you can see what they are writing.

I would like you to work on your own to answer my questions for about ten minutes.

Collaborative small-group work on *Circles and Triangles* (10 minutes)

When students have made a reasonable attempt at the task on their own, organize them into groups of two or three. Give each group a large, fresh piece of paper and a felt-tipped pen. Ask students to have another go at the task, but this time ask them to combine their ideas and make a poster to show their solutions.

Put your own work aside until later in the lesson. I want you to work in groups now.

Your task is to work together to produce a solution that is better than your individual solutions.

While students work in small groups you have two tasks, to note their different approaches to the task, and to support their reasoning.

Note different student approaches to the task

What mathematics do students choose to use? Have they moved on from the mathematical choices made in the assessment task? Do they measure the lengths of the sides of the triangles? Do they draw construction lines? Do they use similar triangles? Do they use algebra? Do they use proportion?

Do students attempt to use the special ratios for 30°, 60°, 90° triangles ($1 : \sqrt{3} : 2$)? If so, how do they do this?

When finding the ratio of the areas of the two triangles, do they find the ratio of the squares of the bases or do they use an alternative method? When finding the ratio of the areas of the two circles, do students find the ratio of the squares of the radii or do they use an alternative method?

Do students fully explain their solutions?

Note any errors, and think about your understanding of students' strengths and weaknesses from the assessment task. You can use this information to focus whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. Focus on supporting students' strategies rather than finding the numerical solution. You may find the questions on the previous page helpful.

If the whole class is struggling on the same issue, write relevant questions on the board.

You may find that some students think the empirical approach (measuring the diagram) is best.

Will your answer change if you measure in inches rather than millimeters?

This question may focus students' attention on the lack of units of measure in the solution and the problem of accuracy.

What are the strengths/weaknesses of this approach?

Are your measurements exact?

Do you think that, if we asked another group that used this same method, they would come up with exactly the same answer as you?

Collaborative small-group analysis of *Sample Responses to Discuss* (20 minutes)

When students have had sufficient time to attempt the problem in their group, give each group copies of the *Sample Responses to Discuss*. This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy.

You may decide there is not enough time for each group to work through all four pieces of work. In that case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at different approaches. Other groups, that have struggled with a particular approach may benefit from seeing a student version of the same strategy.

Here are some different solutions to the problem.

Compare these solutions with your own.

Imagine you are the teacher. Describe how the student approached the problem.

Write your explanation on each solution.

What do you like/dislike about the work?

What isn't clear about the work?

What questions would you like to ask this student?

To encourage students to do more than check to see if the answer is correct, you may wish to use the projector resource *Analyzing Sample Responses to Discuss*. During the group work, check to see which of the explanations students find more difficult to understand.

Plenary whole-class discussion: comparing different solution methods (15 minutes)

Organize a whole-class discussion comparing the four given solutions. Collect comments and ask for explanations.

We are going to look at and compare the four solutions.

Can you explain Bill's method?

Why does Carla draw another triangle in the inner circle?

Encourage students to challenge explanations while keeping your own interventions to a minimum.

Do you agree with Tyler's explanation? [If yes] Explain again, in your own words. [If no] Explain what you think, then.

Finally, ask students to evaluate and compare methods.

Which one did you like best? Why?

Which approach did you find most difficult to understand? Why?

Did anyone come up with a method different from these?

Some issues that might be discussed, with suggested questions and prompts, are given below.

Anya uses measurement

Strengths: It is easy to do. It gives you a feeling for the answer. Anya's calculations are correct. She has rounded to two decimal places.

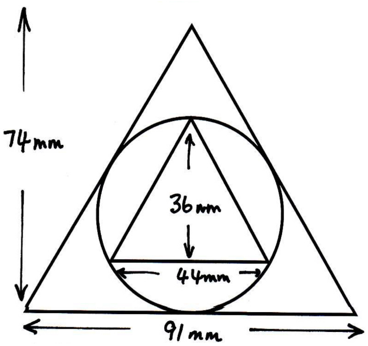
Weaknesses: You only know it is true for the particular case you measure. It's not exact. It doesn't tell you why it's true. Anya does not calculate the areas of the circles, or their ratio (About: $25^2 : 11^2 = 5 : 1$).

Do you think Anya's answer is accurate?

Would an answer rounded to four decimal places be better?

What do you think the answer should be?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.



1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

$\Delta = \frac{1}{2} \cdot 74 \cdot 91 = 3367$

$\Delta = \frac{1}{2} \cdot 36 \cdot 44 = 792$

Ratio $= \frac{3367}{792} = 4 \cdot 25$

Bill uses algebra and ratios

Strengths: Bill's method does not depend on the size of the diagram.
You can use this method for all sorts of problems.

Weaknesses: Bill's work is difficult to follow. There are gaps in his explanation, and it is quite difficult work. Bill does not answer the question, as he does not calculate the ratio of the areas of the triangles.

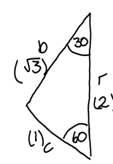
He does not explain why the side lengths in the triangle are in the ratios he writes down, which is based on these trigonometric ratios:

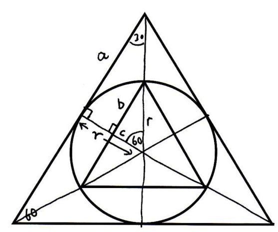
This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

$\frac{c}{r} = \frac{1}{2}$ $\quad c = \frac{r}{2}$

$\frac{b}{r} = \frac{\sqrt{3}}{2}$ $\quad b = \frac{r\sqrt{3}}{2}$

$a = r\sqrt{3}$





1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

$$\sin 30° = \frac{c}{r} = \frac{1}{2}$$

$$\cos 30° = \frac{b}{r} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{a}{r} = \sqrt{3}$$

You could ask students to explain where the ratios in Bill's solution come from, and then to use the lengths to complete the solution.

Why does $\frac{c}{r} = \frac{1}{2}$?

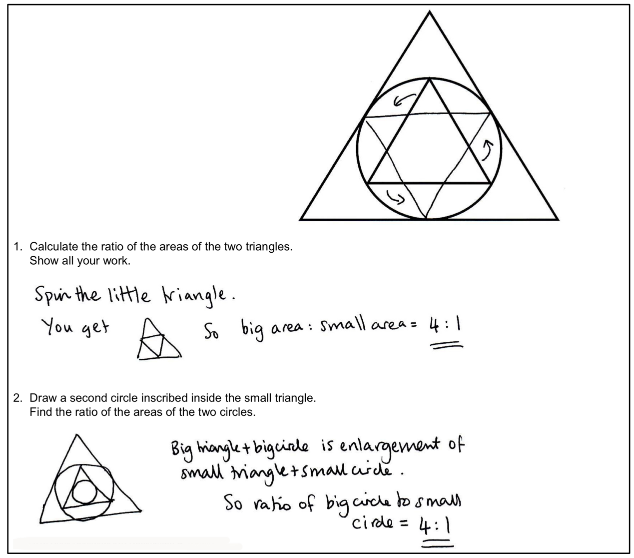
Why does $\frac{b}{r} = \frac{\sqrt{3}}{2}$?

Why does $a = r\sqrt{3}$?

Why does Bill multiply by 6?

What is the ratio of the areas of the two triangles?

Carla uses transformations – rotation and enlargement



Strengths: It is simple. It is clear, even elegant. It is easy to do.

Weaknesses: You have to see it! There are some gaps in the explanation that need to be completed.

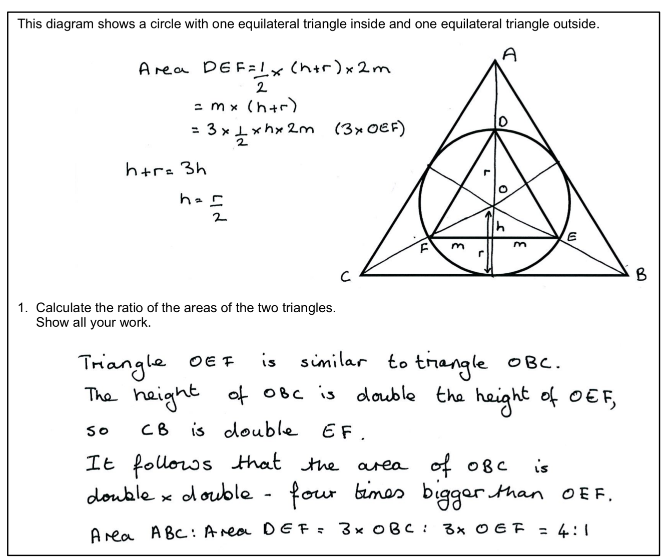
How do you know that, if you rotate the small triangle, it hits the midpoints of the large triangle?

How do you know the four small triangles are congruent?

How do you know the four small triangles are equilateral?

How do you know the circle has been enlarged in the same ratio as the triangle?

Darren uses algebra and similar triangles



Strengths: Darren's method does not depend on the size of the diagram. Darren has labeled the diagram: this makes his work easier to understand.

Weaknesses: Darren's work is difficult to follow at times. He has failed to explain part of his work.

Are triangles OBC and OEF similar? How do you know?

What does Darren mean by "double × double"?

Can you use math to show Darren's answer is correct?

Through comparing different methods, students may come to realize the power of using different methods to solve the same problem.

Next lesson: review individual solutions to *Circles and Triangles* (10 minutes)

Ask students to look again at their original individual solutions to the problem.

Read through your original solution to the Circles and Triangles problem.

Write what you have learned during the lesson.

Suppose a friend began work on this task tomorrow. What advice would you give your friend to help him or her produce a good solution?

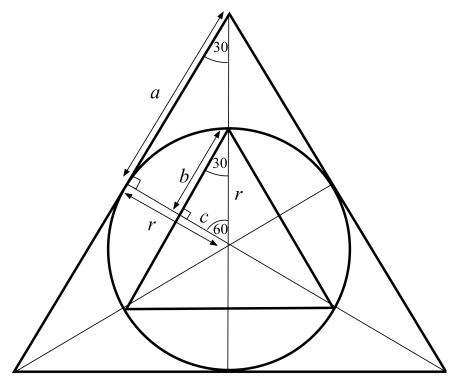
Some teachers set this task as homework.

SOLUTIONS

Circles and Triangles

Bill's method:

Question 1



$$\cos 30° = \frac{b}{r} = \frac{\sqrt{3}}{2} \qquad \Rightarrow b = \frac{r\sqrt{3}}{2}$$

$$\sin 30° = \frac{c}{r} = \frac{1}{2} \qquad \Rightarrow c = \frac{r}{2}$$

$$\tan 60° = \frac{a}{r} = \sqrt{3} \qquad \Rightarrow a = r\sqrt{3}$$

Area of small equilateral triangle:

$$6 \times \frac{1}{2} \times b \times c = 3 \times \frac{r\sqrt{3}}{2} \times \frac{r}{2} = \frac{3\sqrt{3}r^2}{4}$$

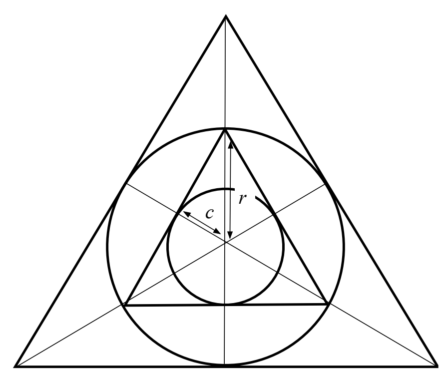
Area of large equilateral triangle:

$$6 \times \frac{1}{2} \times a \times r = 3 \times r\sqrt{3} \times r = 3\sqrt{3}r^2$$

Ratio of area of the outer to the area of the inner triangle =

$$3\sqrt{3}r^2 : \frac{3\sqrt{3}}{4}r^2 = 4:1.$$

Question 2



c is the radius of the inscribed circle. $c = \dfrac{r}{2}$

Ratio of area of circles:

$$= \pi r^2 : \pi c^2$$

$$= \pi r^2 : \pi \frac{r^2}{4}$$

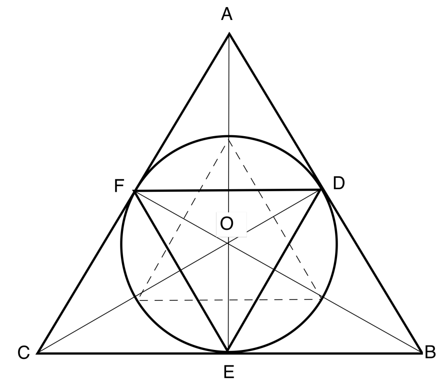
$$= 4:1.$$

Carla's method

Question 1

The small equilateral triangle is rotated through 60° about O, the center of the circle. The arm of the rotation is the radius of the circle. Therefore points D, E, and F are all points on the circumference of the circle.

These points bisect the sides of Δ ABC.



Δ CFE is isosceles (CF = CE because the lengths of two tangents to a circle from a point are equal), so ∠CFE = ∠FEC = (180 − 60) ÷ 2 = 60°.
Therefore Δ CFE is equilateral.

It follows by symmetry that all four small triangles are equilateral and congruent.

Hence the ratio of the area of the outer to the area of the inner triangle = 4 : 1.

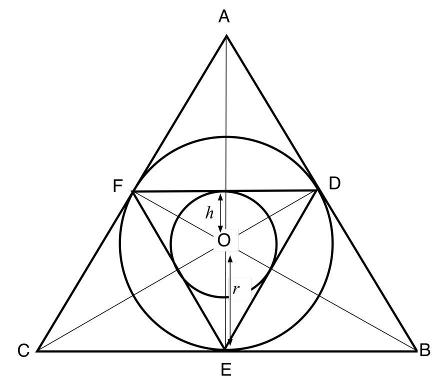
Question 2

Ratio of the area of the outer to the area of the inner triangle:

= (3 × area Δ OCB) : (3 × area Δ FDO)

$$=\left(3\times\frac{1}{2}\times r\times CB\right) : \left(3\times\frac{1}{2}\times h\times\frac{1}{2}\times CB\right)$$

$$=2r \ : \ h.$$



Since we know from Q1 that this ratio is 4 : 1 $\Rightarrow h=\frac{r}{2}$.

Ratio of area of circles

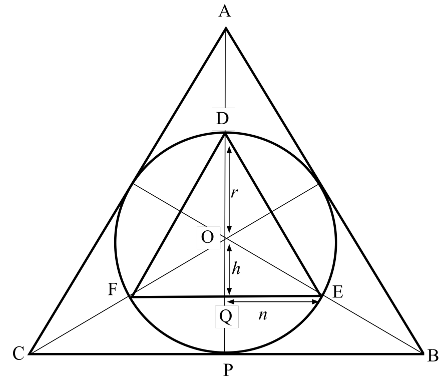
$$=\pi r^2 \ : \ \pi h^2$$

$$=\pi r^2 : \pi\frac{r^2}{4}$$

$$=4 \ : \ 1.$$

Darren's method

Question 1



Area of Δ DEF = 3 × area of Δ OEF

$$\Rightarrow \frac{1}{2} \times 2n \times (h + r) = 3 \times \frac{1}{2} \times 2n \times h$$

$$\Rightarrow n(h + r) = 3nh$$

$$\Rightarrow h + r = 3h$$

$$\Rightarrow h = \frac{r}{2}$$

Triangle OQE is similar to triangle OPB:

∠POB is common to both triangles and OQE = OPB = 90° (altitudes of an equilateral triangle).

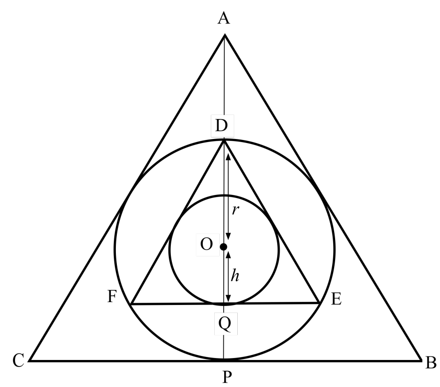
Therefore PB = $2n$ and so CB = $4n$ (altitudes of an equilateral triangle bisect a side).

Area Δ ABC = 3 x area Δ OBC $= 3 \times \frac{1}{2} \times 4n \times r = 6nr$.

Area Δ DEF = 3 x area Δ OEF $= 3 \times \frac{1}{2} \times 2n \times \frac{r}{2} = \frac{3nr}{2}$.

Ratio of areas of triangles $= 6nr : \frac{3nr}{2} = 4 : 1$.

Question 2



h is the radius of the inscribed circle.

$h = \frac{r}{2}$.

The ratio of the area of the outer circle to the area of the inner circle:

$= \pi r^2 \ : \ \pi h^2$

$= \pi r^2 \ : \ \pi \frac{r^2}{4}$

$= 4 \ : \ 1$.

Circles and Triangles

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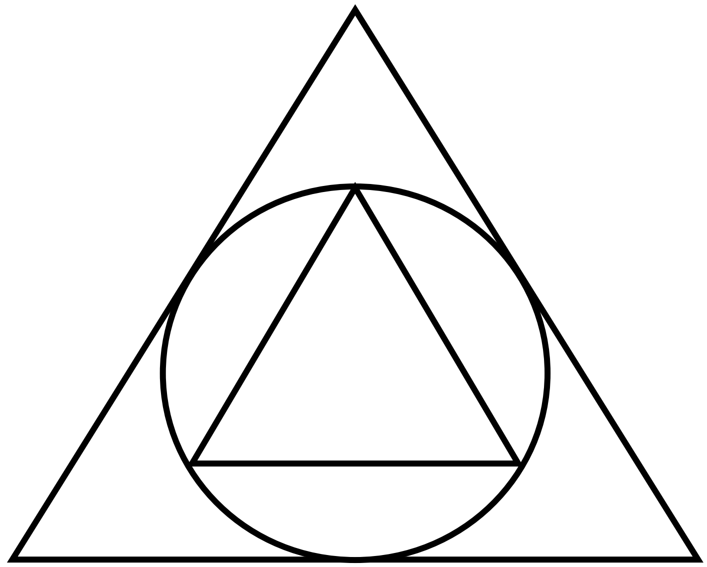


1. Calculate the *exact* ratio of the areas of the two triangles.
   Show all your work.

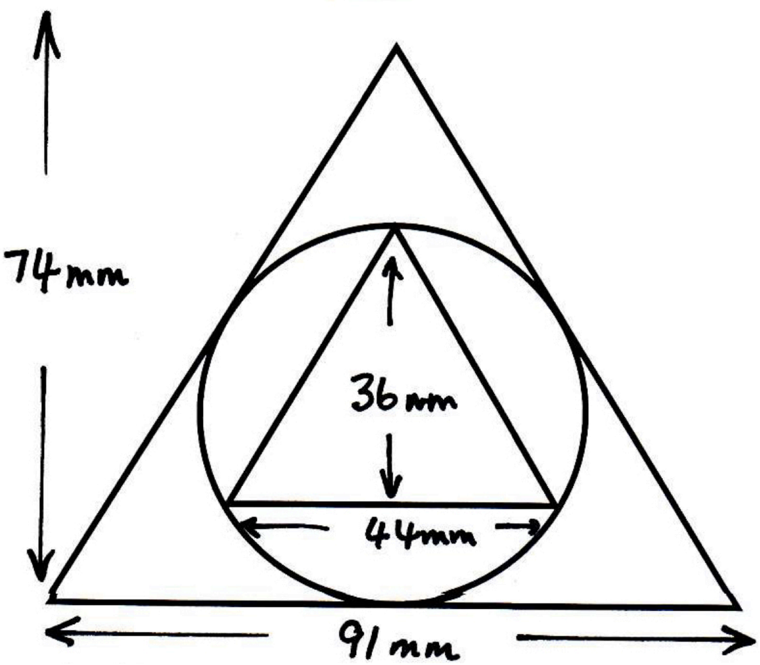
2. Draw a second circle inscribed inside the small triangle.
   Find the *exact* ratio of the areas of the two circles.

Sample Responses to Discuss: Anya

Imagine you are Anya's teacher. Describe how Anya approached the problem.

Write your explanation on a separate sheet.
What do you like/dislike about the work?
What is unclear about the work?
What questions would you like to ask Anya?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.



1. Calculate the ratio of the areas of the two triangles.
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$$\Delta = \frac{1}{2} \cdot 74 \cdot 91 = 3367$$

$$\Delta = \frac{1}{2} \cdot 36 \cdot 44 = 792$$

$$\text{Ratio} = \frac{3367}{792} = 4 \cdot 25$$

Sample Responses to Discuss: Bill

Imagine you are Bill's teacher. Describe how Bill approached the problem.

Write your explanation on a separate sheet.
What do you like/dislike about the work?
What is unclear about the work?
What questions would you like to ask Bill?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

$\frac{c}{r} = \frac{1}{2}$ $\quad c = \frac{r}{2}$

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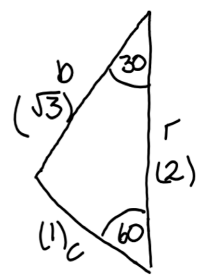
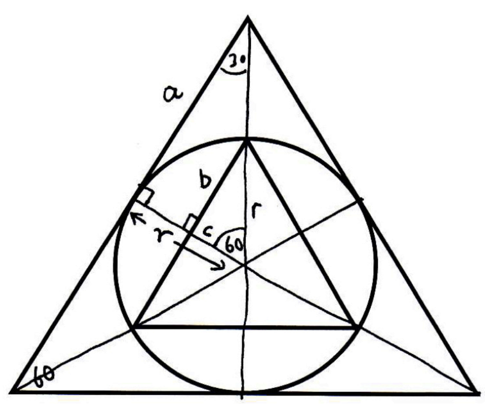




1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Area of small triangle $= 6 \times \frac{1}{2} \times b \times c = 6 \times \frac{1}{2} \times \frac{r\sqrt{3}}{2} \times \frac{r}{2} = \frac{3r^2\sqrt{3}}{4}$

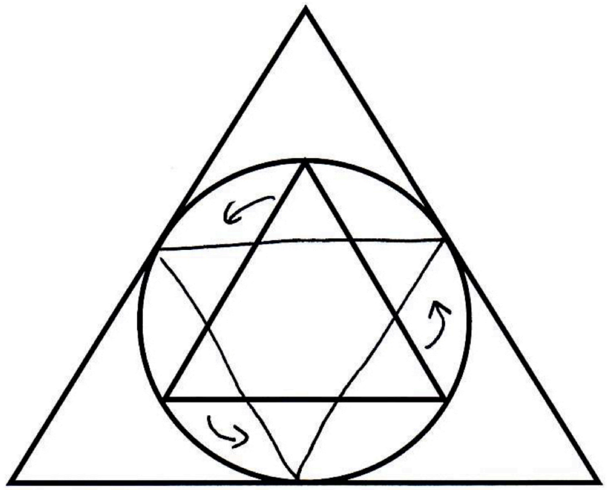
Area of large triangle $= 6 \times \frac{1}{2} \times a \times r = 3 \times r\sqrt{3} \times r = 3r^2\sqrt{3}$

Sample Responses to Discuss: Carla

Imagine you are Carla's teacher. Describe how Carla approached the problem.

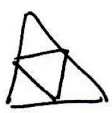
Write your explanation on a separate sheet.
What do you like/dislike about the work?
What isn't clear about the work?
What questions would you like to ask Carla?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

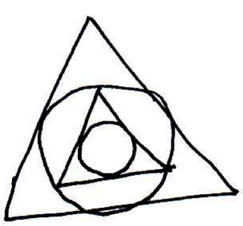


1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Spin the little triangle.

You get  So big area : small area = 4 : 1

2. Draw a second circle inscribed inside the small triangle.
   Find the ratio of the areas of the two circles.



Big triangle + big circle is enlargement of small triangle + small circle.

So ratio of big circle to small circle = 4 : 1

Sample Responses to Discuss: Darren

Imagine you are Darren's teacher. Describe how Darren approached the problem.

Write your explanation on a separate sheet.
What do you like/dislike about the work?
What is unclear about the work?
What questions would you like to ask Darren?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

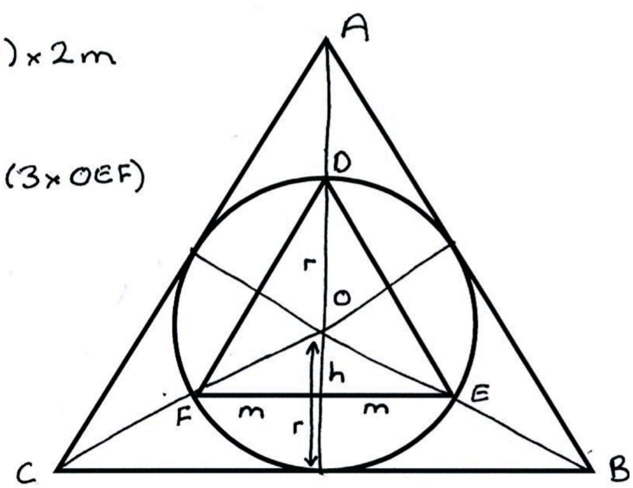
Area DEF $= \frac{1}{2} \times (h+r) \times 2m$

$= m \times (h+r)$

$= 3 \times \frac{1}{2} \times h \times 2m$ $(3 \times OEF)$

$h + r = 3h$

$h = \frac{r}{2}$



1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Triangle OEF is similar to triangle OBC.
The height of OBC is double the height of OEF,
so CB is double EF.
It follows that the area of OBC is
double × double - four times bigger than OEF.

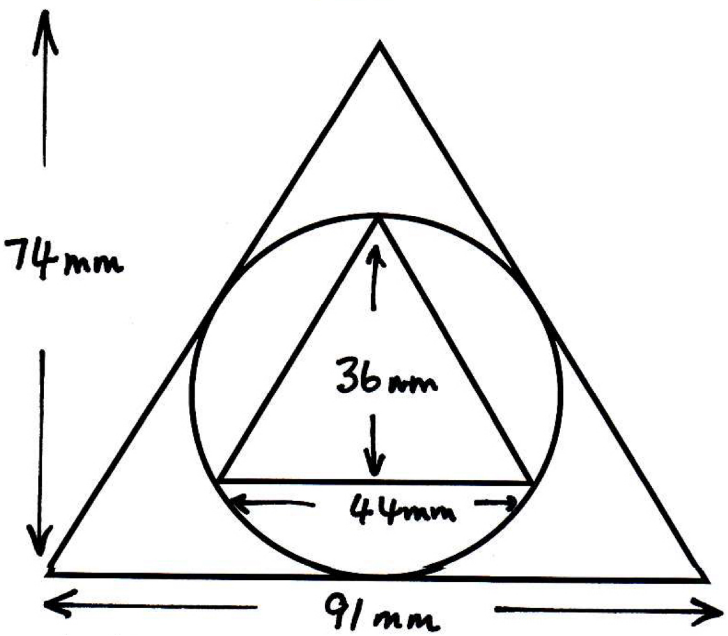
Area ABC : Area DEF = $3 \times OBC : 3 \times OEF = 4:1$

Analyzing Sample Responses to Discuss

- Explain what the student has done.
- What do you like/dislike about the work?
- What is unclear about the work?
- What questions would you like to ask this student?

Anya's Solution

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.



1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

$$\Delta = \frac{1}{2} \cdot 74 . 91 = 3367$$

$$\Delta = \frac{1}{2} \cdot 36 . 44 = 792$$

$$\text{Ratio} = \frac{3367}{792} = 4.25$$

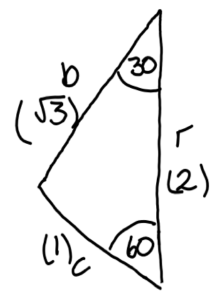
Bill's Solution (1)

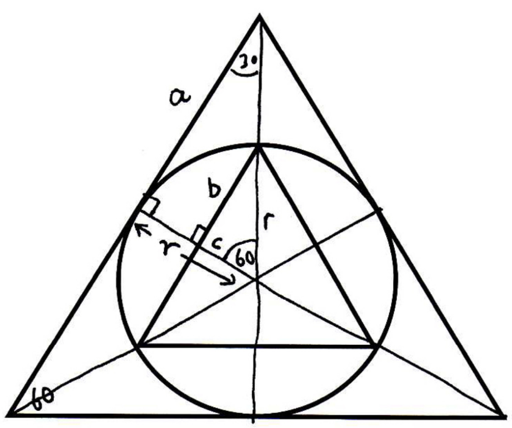
This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

$\frac{c}{r} = \frac{1}{2}$ $\quad c = \frac{r}{2}$

$\frac{b}{r} = \frac{\sqrt{3}}{2}$ $\quad b = \frac{r\sqrt{3}}{2}$

$a = r\sqrt{3}$





1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

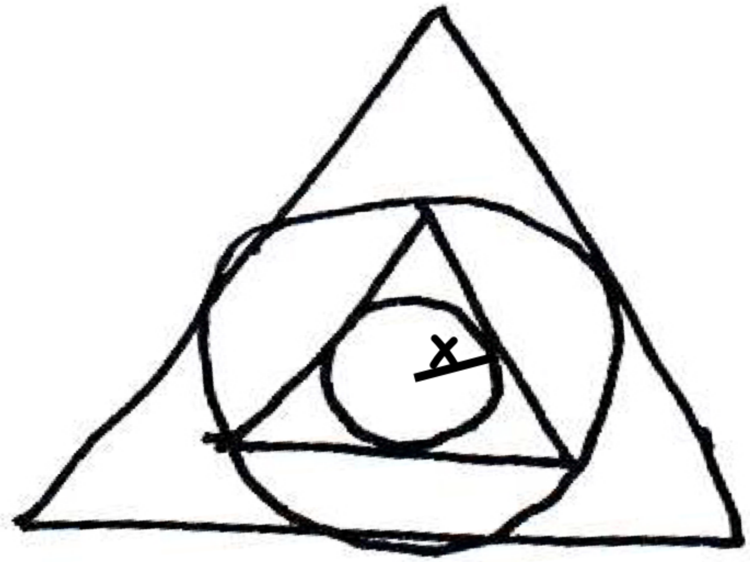
Bill's Solution (2)

2. Draw a second circle inscribed inside the small triangle.
   Find the ratio of the areas of the two circles.

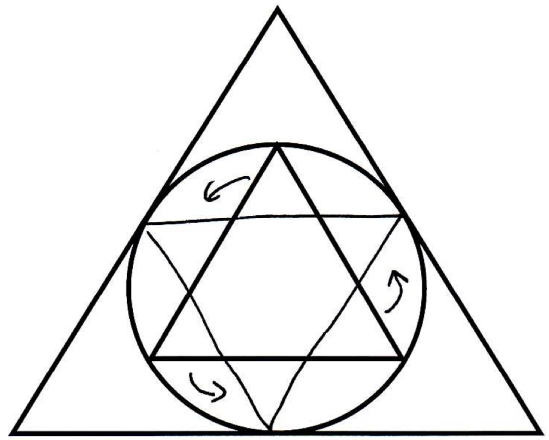
Large circle $= \pi r^2$

Small circle $= \pi x^2 = \pi \left(\frac{r}{2}\right)^2 = \frac{\pi r^2}{4}$

Ratio of areas $= 1 : \frac{1}{4} = \underline{\underline{4:1}}$

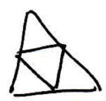


Carla's Solution

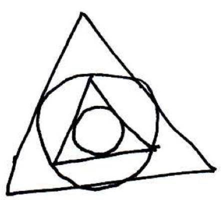


1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Spin the little triangle.

You get  So big area : small area = $\underline{\underline{4 : 1}}$

2. Draw a second circle inscribed inside the small triangle.
   Find the ratio of the areas of the two circles.



Big triangle + big circle is enlargement of small triangle + small circle.

So ratio of big circle to small circle = $\underline{\underline{4 : 1}}$

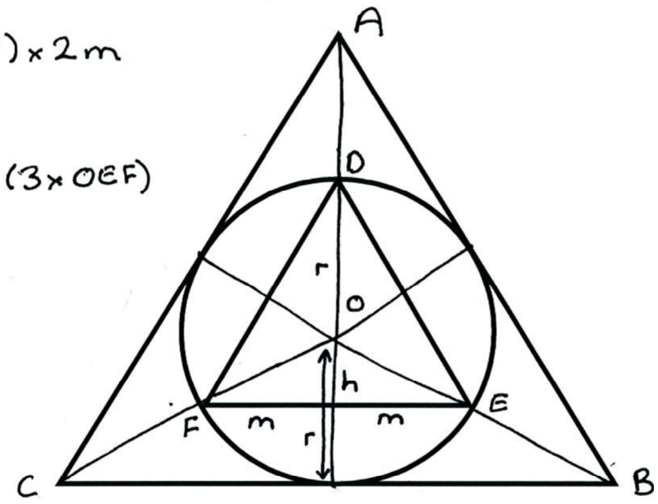
Darren's Solution

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

$$\text{Area } DEF = \frac{1}{2} \times (h+r) \times 2m$$
$$= m \times (h+r)$$
$$= 3 \times \frac{1}{2} \times h \times 2m \quad (3 \times OEF)$$

$$h + r = 3h$$

$$h = \frac{r}{2}$$



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Area ABC : Area DEF = 3 × OBC : 3 × OEF = 4 : 1

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

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of these materials in their classrooms, to their students, and to
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